MC/TH 94/10

Constraints on Four-Point Couplings in Low-Energy Meson Interactions

Dimitri Kalafatis and Michael C. Birse

Theoretical Physics Group
Department of Physics and Astronomy
The University of Manchester
Manchester M13 9PL
UNITED KINGDOM

June 1994

Abstract

We investigate from first principles the introduction of isospin-1 vector and axial-vector fields into the nonlinear sigma model. Chiral symmetry is nonlinearly realised and spin-1 fields are assumed to transform homogeneously under chiral rotations. By requiring the Hamiltonian of the theory to be bounded from below we find inequalities relating three- and four-point meson couplings. This leads to a low-energy phenomenological Lagrangian for the nonanomalous sector of $\pi\rho a_1$ strong interactions.

1. Introduction

At low energies strong interactions can be described by an effective Lagrangian in terms of mesons [1]. This should comply with the approximate symmetries of low energy strong interactions such as chiral symmetry. Chiral symmetry provides information about the general structure of the couplings between mesons whereas the coupling constants entering the effective Lagrangian are related to more detailed features of the underlying QCD

dynamics. Unfortunately it is still impossible to extract from QCD the values of the effective low energy parameters. Some of these can be related to phenomenologically known meson observables, like masses and decay widths, but most of the higher order parameters remain unknown.

The starting point for such an effective Lagrangian is the nonlinear sigma model of pseudoscalar pions. This realises spontaneous breaking of chiral symmetry, a central feature of low energy QCD and introduces a single parameter, the pion decay constant. The experimental discovery of meson resonances as well as some theoretical notions such as the large N_c expansion of QCD [2], strongly support the idea of introducing mesons other than the pion into this model. There is a considerable amount of work in the literature treating the role played by massive spin-1 mesons (the ρ- and the a_1-mesons) in low-energy Lagrangians. In most of these works isovector resonances are introduced as massive Yang-Mills particles [3] or as gauge bosons of local chiral symmetry [4]. In these approaches some of the new coupling constants can be determined by fitting to processes like $\rho \to \pi\pi$ or $a_1 \to \rho\pi$. The remaining four-point and three-point coupling constants are then completely determined by the gauge symmetry assumption.

Although these approaches are consistent with the phenomenologically successful notion of vector meson dominance, it should be noted that there is neither experimental evidence nor theoretical prejudice from QCD to support the existence of a gauge symmetry in low energy hadronic interactions. Furthermore as the authors of Ref. [5] have shown vector meson dominance is not a feature unique to the models of Refs. [3-4]. It can also be obtained in models where chiral symmetry is realised in a different manner.

Here we follow a different approach, as suggested by [6], of writing down a general Lagrangian consistent with basic principles of field theory and chiral symmetry. Vector meson dominance can be implemented later, if so desired, by specific choices of parameters. From this point of view it is reasonable to assume a homogeneous transformation law for isovector spin-1 fields, instead of that used in ref. [4]. As was shown recently for the case of the $\pi\rho$ system [6], without making any additional symmetry assumption, constraints relating three- and four-point coupling strengths do exist. These derive from demanding the Hamiltonian to be bounded from below. The results of [6] are encouraging enough to suggest a systematic investigation of four-point couplings in more realistic theories that include the axial-vector meson.

The purpose of this work is therefore to extend the analysis of ref. [6] to the description of interacting pions, ρ- and a_1-mesons assuming that the spin-1 isovector fields transform homogeneously under nonlinear chiral symmetry. The $\pi\rho a_1$ system turns out to be more complicated than the $\pi\rho$ case but it is more interesting since there are both vector and axial-vector mesons with masses of around 1 GeV. In section 2 we define the transformation properties of the fields. Section 3 is devoted to the investigation of the energies of nonperturbative field configurations in the framework of the minimal three-point coupling theory. We show that these energies are unbounded from below. In section 4 we show that the inclusion of four-point effective couplings counterbalances the dangerous contributions of the three-point terms to these energies. We derive inequalities between three- and four-meson couplings for the theory to make sense. In section 5 we discuss how unitarity arguments based on vector dominance could lead to saturation of these inequalities and present a low-energy $\pi\rho a_1$ effective Lagrangian consistent with chiral symmetry and general field theoretical principles.

2. Transformations under chiral rotations

Our starting point is the Lagrangian of the nonlinear sigma model defined in terms of the $SU(2)$ field U as:

$$\mathcal{L}_{NL\sigma} = \frac{f^2}{4} < \partial^\mu U \partial_\mu U^\dagger >, \tag{1}$$

f being the pion decay constant and the symbols "$< >$" denoting a trace in SU(2) space. Since we are interested here in the structure of the theory for large amplitude field configurations we define U as $U = \exp(i\vec{\tau}.\vec{F}(x))$ with the pion field given by $\vec{F} = F\hat{F}$. Other parametrisations are perhaps more suitable for perturbative evaluations of Green' s functions, but are not as convenient for investigations of the large field region.

The Lagrangian of eq. (1) is manifestly invariant under the linear $SU(2)_L \otimes SU(2)_R$ global transformation $U \rightarrow g_L U g_R^\dagger$ with $g_L, g_R \in SU(2)$. It is also invariant under the following nonlinear rotation [7] of the square root u of U:

$$u(\vec{F}) \;\rightarrow\; g_L u(\vec{F}) h^\dagger(\vec{F}) \;=\; h(\vec{F}) u(\vec{F}) g_R^\dagger, \tag{2}$$

$h(\vec{F})$ being an $SU(2)$-matrix that depends nonlinearly on the pion fields. This compensating transformation $h(\vec{F})$ ensures that U transforms linearly.

With the pion unitary matrix transforming as in eq. (2) one defines the following field gradients:

$$\begin{aligned} u_\mu =& i(u^\dagger \partial_\mu u - u \partial_\mu u^\dagger) \\ \Gamma_\mu =& \frac{1}{2}(u^\dagger \partial_\mu u + u \partial_\mu u^\dagger). \end{aligned} \tag{3}$$

The axial-vector and vector characters respectively of u_μ and Γ_μ are manifest from their expressions in terms of pseudoscalar pion fields $\vec{F}$:

$$\begin{aligned} u_\mu =& -\tau_k \left[\hat{F}_k \hat{F}_m + \frac{\sin F}{F}(\delta_{km} - \hat{F}_k \hat{F}_m)\right] \partial_\mu \vec{F}_m \\ \Gamma_\mu =& i\vec{\tau} \, . \, (\vec{F} \times \partial_\mu \vec{F}) \, \frac{\sin^2(F/2)}{F^2}. \end{aligned} \tag{4}$$

From eq. (2) the transformations of these gradients under chiral symmetry are given by:

$$\begin{aligned} u_\mu &\to h(\vec{F}) u_\mu h^\dagger(\vec{F}) \\ \Gamma_\mu &\to h(\vec{F}) \Gamma_\mu h^\dagger(\vec{F}) + h(\vec{F}) \partial_\mu h^\dagger(\vec{F}). \end{aligned} \tag{5}$$

The quantity u_μ is seen to transform homogeneously whereas the transformation of Γ_μ contains an inhomogeneous part as a result of the field dependence of $h(\vec{F})$.

In extending this to spin-1 isovector particles, in particular the ρ and the a_1, the immediate question is: how should these fields transform in this framework? Using the matrix h one finds that if these fields are to be described by Lorentz vectors there are only two possibilities forming a group: homogeneous or inhomogeneous.

In the case of inhomogeneous transformation laws [4] the associated lowest order invariant Lagrangian preserves not only chiral symmetry but also a certain sort of a gauge symmetry. Furthermore in the inhomogeneous approach it is impossible to define similar transformations for both the ρ and the a_1 fields, simply because the associated particles have opposite parity.

In contrast the homogeneous transformation is the simplest one consistent with chiral symmetry and has the nice feature that it can be applied to both the ρ and the a_1 fields. We adopt this first-principles point of view and assume that the ρ- and the a_1-mesons transform homogeneously under the nonlinear chiral group:

$$\begin{aligned} V_\mu &\to h(\vec{F}) V_\mu h^\dagger(\vec{F}) \\ A_\mu &\to h(\vec{F}) A_\mu h^\dagger(\vec{F}) \end{aligned} \tag{6}$$

where $V_\mu = \vec{\tau}.\vec{V}_\mu$ and $A_\mu = \vec{\tau}.\vec{A}_\mu$. It is clear that Γ_μ is the necessary ingredient for the definition of covariant derivatives of spin-1 fields transforming as in eq. (6)

$$\nabla_\mu = \partial_\mu + [\Gamma_\mu,\ ]. \tag{7}$$

It is easy to check now that $\nabla_\mu V_\nu$ and $\nabla_\mu A_\nu$ also transform homogeneously: $\nabla_\mu V_\nu \to h \nabla_\mu V_\nu\ h^\dagger$ and similarly $\nabla_\mu A_\nu \to h \nabla_\mu A_\nu\ h^\dagger$.

3. Three-point couplings

With the transformation rules defined previously the invariant Lagrangian at quadratic order in the fields is given by

$$\begin{aligned}\mathcal{L}^{(2)}_{\pi\rho a_1} =& \frac{f^2}{4} < u_\mu u^\mu > - \frac{1}{4} < V_{\mu\nu} V^{\mu\nu} > - \frac{1}{4} < A_{\mu\nu} A^{\mu\nu} > \\ &+ \frac{M_\rho^2}{2} < V_\mu V^\mu > + \frac{M_a^2}{2} < A_\mu A^\mu >,\end{aligned} \tag{8}$$

where $V_{\mu\nu} = \nabla_\mu V_\nu - \nabla_\nu V_\mu$ and $A_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$ are the covariant field strengths of the spin-1 resonances. We introduce chirally invariant mass terms for the ρ- and the a_1-mesons and we assume that the coupling $c < A_\mu u^\mu >$ is not present. This latter coupling is a result of $a_1 - \pi$ mixing which, at lowest order, is certainly not allowed if at some level one is to identify the fields with the physical states. With the choice $c = 0$ no diagonalisation of $\pi\rho a_1$ interactions is needed - obviously not a disadvantage of our framework.

At the three-point level there is a number of possible chirally invariant terms consistent with charge conjugation and parity invariance. Leaving three-point interactions among the vector mesons aside for a future analysis, we consider here chirally invariant three-point couplings with at least one pion field gradient:

$$\begin{aligned}\mathcal{L}^{(3)}_{\pi\rho a_1} = -\frac{i}{2\sqrt{2}} \Big\{ & g_1 < V_{\mu\nu} [u^\mu, u^\nu] > \ + g_2 < A_{\mu\nu} \big([V^\mu, u^\nu] - [V^\nu, u^\mu]\big) > \\ & + g_3 < V_{\mu\nu} \big([A^\mu, u^\nu] - [A^\nu, u^\mu]\big) > \Big\}.\end{aligned} \tag{9}$$

The Lagrangian $\mathcal{L}^{(2)}_{\pi\rho a_1} + \mathcal{L}^{(3)}_{\pi\rho a_1}$ has six free parameters that can be determined by fitting to low energy meson observables like masses, decay widths etc. In principle one would like to

determine these parameters from QCD but, while some recent investigations in the ENJL model [8] suggest that the problem is not hopeless, a sensible method to perform such an extraction from QCD has not yet been discovered.

The issue we address here is rather different: assuming that g_1, g_2, g_3 are somehow given by the underlying QCD dynamics, are there any relations between these parameters and higher order ones? The results of ref. [6] suggest that this question should be addressed in a nonperturbative framework. In particular does the theory defined by equations (8, 9) yield a Hamiltonian that is bounded from below? To find an answer we study the effect of three-point interactions in the classical sector of the theory. We construct the Hamiltonian associated with the Lagrangian $\mathcal{L}^{(2)}_{\pi\rho a_1} + \mathcal{L}^{(3)}_{\pi\rho a_1}$ in terms of the canonical degrees of freedom: the fields $\vec{F},\ \vec{V}_i,\ \vec{A}_i$ and their conjugate momenta, respectively $\vec{\phi},\ \vec{\pi}_i, \vec{\chi}_i$. The Hamiltonian functional can be written as a sum of two terms $H = H_T + H_V$, where the kinetic energy is H_T and the potential energy is H_V. The potential part contains only space components and in the three-point case is given by

$$H_V = \int d^3x \bigg\{ \frac{f^2}{2}(u_i)^2_k + M_a^2 (A_i)^2_k + \frac{1}{2}(A_{ij})_k \big[\ A_{ij} + i\sqrt{2} g_2([V_i, u_j] - [V_j, u_i])\ \big]_k \\ + M_\rho^2 (V_i)^2_k + \frac{1}{2}(V_{ij})_k \big[\ V_{ij} + i\sqrt{2} g_1 [u_i, u_j] + i\sqrt{2} g_3([A_i, u_j] - [A_j, u_i])\ \big]_k \bigg\}. \tag{10}$$

The kinetic piece needs some work in order to eliminate the dependent variables $\vec{V}_0,\ \vec{A}_0$. A detailed derivation of it is given in the Appendix; here we simply state the result:

$$H_T = \int d^3x \bigg\{ \frac{1}{2}\vec{\Phi}\mathcal{A}^{-1}\vec{\Phi} + \frac{\vec{\pi}_i^2}{4} + \frac{\vec{\chi}_i^2}{4} + \frac{1}{2}\vec{\Gamma}\mathcal{P}^{-1}\vec{\Gamma} \bigg\}, \tag{11}$$

where $\vec{\Phi},\ \vec{\Gamma}$ are linearly related to the momenta $\vec{\phi},\ \vec{\pi}_i,\ \vec{\chi}_i$ and $\mathcal{A},\ \mathcal{P}$ are isospin tensor functions of $\vec{F},\ \vec{V}_i,\ \vec{A}_i$. The rather lengthy expressions for these objects are also given in the Appendix.

In order to exhibit the problematic structure of the theory defined by eq. (9) we investigate the energy of a classical coherent configuration of the meson fields. The simplest such object one can imagine has an isospin content specified by a constant unit vector $\hat{F}$:

$$\vec{F}_0(\vec{x}) = F(\vec{x})\ \hat{F}, \tag{12}$$

where $F(\vec{x})$ is a regular function of space. Such a configuration is topologically trivial and carries no baryon number. For the vector and the axial vector fields it is convenient to assume that they are parallel to the pion field:

$$\begin{aligned} \vec{V}_i(\vec{x}) =& V_i(\vec{x})\ \hat{F} \\ \vec{A}_i(\vec{x}) =& A_i(\vec{x})\ \hat{F}. \end{aligned} \tag{13}$$

It is clear from the definitions in eqs. (12-13) that all commutators between space components including the connection Γ_i vanish. As a result the potential energy is simply given by:

$$H_V = \int d^3x \left\{ \frac{f^2}{2}(\partial_i F)^2 + M_a^2 A_i^2 + \frac{1}{2}(\partial_i A_j - \partial_j A_i)^2 + M_\rho^2 V_i^2 + \frac{1}{2}(\partial_i V_j - \partial_j V_i)^2 \right\}, \tag{14}$$

the potential energy of a free theory as if F was a massless scalar field and V_i and A_i were the space components of two massive spin-1 mesons. The important feature for our investigation is that this potential energy is positive and does not depend on the couplings of the theory. We therefore concentrate in the kinetic energy of the theory as given by H_T.

To evaluate the kinetic energy for our meson state we need the explicit structure of matrices $\mathcal{A}$ and $\mathcal{P}$ (see the Appendix) in terms of the fields $F,\ V_i,\ A_i$. Because of the particular isospin structure we consider here all these matrices can be simply decomposed in terms of two symmetric isospin tensors: the unit tensor and $\hat{F} \otimes \hat{F}$. As a consequence only momenta that point in a direction perpendicular to that of the pion actually "see" the couplings to the vector mesons. We assume the following forms:

$$\begin{aligned} \vec{\phi} =& \phi(\vec{x})\ \hat{\phi} \\ \vec{\pi}_i =& \pi_i(\vec{x})\ \hat{F} \\ \vec{\chi}_i =& \chi_i(\vec{x})\ \hat{F}, \end{aligned} \tag{15}$$

with $\hat{\phi} \cdot \hat{F} = 0$. Using now the forms of eqs. (12, 13, 15) in (11) we find the kinetic energy of our field configuration

$$H_T = \int d^3x \left\{ \frac{\phi^2}{2f^2 s^2 \mathcal{I}} + \frac{1}{4}\left[\pi_i^2 + \chi_i^2 + \frac{(\partial_i \pi_i)^2}{M_\rho^2} + \frac{(\partial_i \chi_i)^2}{M_a^2} \right] \right\}, \tag{16}$$

where s is a shorthand notation for $\sin F/F$.

The crucial feature here is the structure of the dimensionless "inertial" parameter $\mathcal{I}$ which contains all the nontrivial effects due to the inclusion of massive spin-1 fields:

$$\mathcal{I} = \frac{1}{f^4 M_1 M_2}\Big(f^4 M_1 M_2 - 8(g_2^2 V_i^2 M_2 M_\rho^2 + (g_1\partial_i F - g_3 A_i)^2 M_1 M_a^2) \\ + 16\big[g_2^4 M_2(V_i^2(\partial_i F)^2 - (V_i\partial_i F)^2) + g_3^4 M_1(A_i^2(\partial_i F)^2 - (A_i\partial_i F)^2)\big]\Big), \tag{17}$$

with $M_1 = (1/f^2)\big[2M_\rho^2 - 4g_2^2(\partial_i F)^2\big]$ and $M_2 = (1/f^2)\big[2M_a^2 - 4g_3^2(\partial_i F)^2\big]$. While in the case of the nonlinear sigma model with vanishing couplings $g_1,\ g_2,\ g_3$ the "inertial" parameter $\mathcal{I}$ is simply equal to 1, here it acquires negative contributions from the vector and the axial-vector fields. For very small fields $F,\ V_i,\ A_i \approx 0$ appropriate to perturbation theory one has $\mathcal{I} \approx 1$ so the problem does not appear in perturbative expansions of scattering amplitudes.

For nonperturbative configurations the situation changes dramatically since then negative contributions proportional to quadratic powers of the couplings can drive $\mathcal{I}$ to zero or negative values. The Hamiltonian density acquires poles and the energy is not bounded from below. To give an idea of the energetic scales where such troubles arise let us consider a localised meson wave carrying momentum k_i and of amplitude $F \approx 1$. As a further simplification of our original ansatz we assume that all classical fields vanish except F and ϕ. The gradient $\partial_i F$ is roughly approximated by k_i and $\mathcal{I}$ inside the meson wave looks like:

$$\mathcal{I} \approx \frac{1 - 2\left(\dfrac{g_3^2}{M_a^2} + 2\dfrac{g_1^2}{f^2}\right)k^2}{1 - 2\dfrac{g_3^2}{M_a^2}k^2}. \tag{18}$$

At small or very large momenta k the inertial parameter is positive since the denominator and numerator in eq. (18) then have the same sign. But for k^2 in the intermediate range

$$\left(2\frac{g_3^2}{M_a^2} + 4\frac{g_1^2}{f^2}\right)^{-1} < k^2 < \frac{M_a^2}{2g_3^2} \tag{19}$$

$\mathcal{I}$ becomes negative and as a consequence *the kinetic energy density is negative*, making the theory ill defined in these regions.

Taking reasonable numerical values of the coupling constants [9], the region of dangerous momenta is found to be 0.4 GeV $< k <$ 2.0 GeV, which includes the range of masses of the ρ and the a_1 resonances. However this is precisely the range that one would like to describe by extending the low-energy effective theories to include spin-1 mesons. Notice also that if one switches off the couplings to the axial-vector meson, $g_3 = g_2 = 0$, the region where $\mathcal{I}$ becomes negative is modified to $f^2/(4g_1^2) < k^2 < \infty$. This simple example shows that although inclusion of the a_1-meson reduces the chance of the kinetic energy of the theory becoming negative, it is not able to cure the pathologies of the theory at the three-point level.

In general then the Hamiltonian associated with the simplest three-point $\pi\rho a_1$ interactions *is not bounded from below* which is of course unacceptable. This extends the results of reference [6] where the $\pi\rho$ system was studied and where the energy of a topologically nontrivial configuration was found to be unbounded from below when one includes only three-meson couplings.

4. Four-point couplings.

In order to cure the pathologies of the Lagrangian (9) we need to consider higher-order terms. As we have seen, troubles emerge because the derivative nature of vector-meson interactions produces singularities in the kinetic part of the Hamiltonian. This suggests that one should analyse the role of four-point couplings that are quadratic in time derivatives of the pion field. The most general chiral Lagrangian at quartic order satisfying C and P invariance, and leading to a Hamiltonian that is at most quadratic in the momenta is:

$$\begin{aligned}\mathcal{L}^{(4)}_{\pi\rho a_1} =& \frac{1}{8}\Big\{ g_4 < [u_\mu, u_\nu]^2 > + 2g_5 < [u_\mu, u_\nu][A^\mu, u^\nu] > + 2g_6\big(< [V_\mu, u_\nu]^2 > \\ & - < [V_\mu, u_\nu][V^\nu, u^\mu] > \big) + 2g_7\big(< [A_\mu, u_\nu]^2 > - < [A_\mu, u_\nu][A^\nu, u^\mu] > \big)\Big\},\end{aligned} \tag{20}$$

where we have introduced four new coupling constants $g_4,\ g_5,\ g_6,\ g_7$. Amongst these terms one can recognise a local four-point pion vertex, the so-called "Skyrme term", as well as $\rho\rho\pi\pi$ and $a_1 a_1 \pi\pi$ vertices and a term contributing to the decay $a_1 \to \pi\pi\pi$.

The potential energy $\tilde{H}_V$ of the theory reads now:

$$\tilde{H}_V = H_V - \frac{1}{4}\int d^3x \Big\{ g_4[u_i,u_j]_k^2 + 2g_5[u_i,u_j]_k[A_i,u_j]_k + 2g_6([V_i,u_j]_k^2 - [V_i,u_j]_k[V_j,u_i]_k) + 2g_7([A_i,u_j]_k^2 - [A_i,u_j]_k[A_j,u_i]_k) \Big\}, \tag{21}$$

where H_V is the functional given by eq. (10), and we use the tilde to denote the corresponding quantity with four-point couplings included. The functional form of the kinetic term, eq. (11), is not modified by the inclusion of four-point couplings: these are entirely contained in the corresponding $\tilde{\vec{\Gamma}}$, $\tilde{\mathcal{P}}$ and $\tilde{\mathcal{A}}$. The expressions for these quantities can be obtained from those at the three-point level by the replacements $g_1^2 \to g_1^2 - g_4,\ g_2^2 \to g_2^2 - g_6,\ g_3^2 \to g_3^2 - g_7$ and $g_1g_3 \to g_1g_3 - \frac{g_5}{2}$.

Turning to the energy of the charge-zero meson configuration defined in the previous section, we note first that its potential energy is unaffected by the new couplings and is still given by eq. (14). The kinetic piece has the same form as in eq. (16) but with a new inertial function, $\tilde{\mathcal{I}}$. After a tedious but straightforward calculation one finds:

$$\begin{aligned}\tilde{\mathcal{I}} = \frac{1}{f^4\tilde{M}_1\tilde{M}_2}\Big\{ & f^4\tilde{M}_1\tilde{M}_2 - 8(g_2^2-g_6)V_i^2M_\rho^2\tilde{M}_2 \\ & + 16\Big[(g_2^2-g_6)^2\big(V_i^2(\partial_iF)^2 - (\partial_iFV_i)^2\big)\Big]\tilde{M}_2 \\ & - 8\Big[(g_1^2-g_4)(\partial_iF)^2 + (g_3^2-g_7)A_i^2 - 2\left(g_3g_1-\frac{g_5}{2}\right)(\partial_iFA_i)\Big]M_a^2\tilde{M}_1 \\ & + 16\Big[\Big((g_1^2-g_4)(g_3^2-g_7) - \left(g_3g_1-\frac{g_5}{2}\right)^2\Big)(\partial_iF)^4 \\ & \quad + (g_3^2-g_7)^2\big(A_i^2(\partial_iF)^2 - (A_i\partial_iF)^2\big)\Big]\tilde{M}_1\Big\},\end{aligned} \tag{22}$$

with $\tilde{M}_1 = (1/f^2)\big[2M_\rho^2 - 4(g_2^2-g_6)(\partial_iF)^2\big]$ and $\tilde{M}_2 = (1/f^2)\big[2M_a^2 - 4(g_3^2-g_7)(\partial_iF)^2\big]$.

We are now in a position to find constraints on the couplings by requiring that for any value of the classical profiles $\partial_iF,\ V_i,\ A_i$ the function $\tilde{\mathcal{I}}$ is non-negative. To do this we consider three simplifying cases where some of the fields vanish. These and their corresponding forms for $\tilde{\mathcal{I}}$ are as follows:

$a)\quad \partial_iF = A_i = 0 \qquad \Rightarrow \qquad \tilde{\mathcal{I}}_a = 1 - \frac{4}{f^2}(g_2^2-g_6)V_i^2$

$$b)\quad \partial_i F = V_i = 0 \qquad \Rightarrow \qquad \tilde{\mathcal{I}}_b = 1 - \frac{4}{f^2}(g_3^2 - g_7)A_i^2 \tag{23}$$

$$c)\quad V_i = A_i = 0 \qquad \Rightarrow \qquad \tilde{\mathcal{I}}_c = \frac{1}{f^2\tilde{M}_2}\Big[2M_a^2 - \big[4(g_3^2 - g_7) + 8(g_1^2 - g_4)\frac{M_a^2}{f^2}\big](\partial_i F)^2 + \frac{16}{f^2}\big[(g_1^2 - g_4)(g_3^2 - g_7) - (g_3 g_1 - \frac{g_5}{2})^2\big](\partial_i F)^4\Big].$$

Requiring positiveness of $\tilde{\mathcal{I}}_{a,b,c}$ for all possible values of the fields leads to the following constraints on the couplings constants:

$$\begin{aligned} g_4 &\geq g_1^2 \\ g_6 &\geq g_2^2 \\ g_7 &\geq g_3^2 \\ (g_1^2 - g_4)(g_3^2 - g_7) &\geq (g_3 g_1 - \frac{g_5}{2})^2. \end{aligned} \tag{24}$$

The second and third inequalities follow immediately from requiring $\mathcal{I}_{a,b}$ to be positive definite. They imply that both $\tilde{\mathcal{M}}_1$ and $\tilde{\mathcal{M}}_2$ are also positive definite. For large amplitude fields the quartic power $(\partial_i F)^4$ dominates over the quadratic one $(\partial_i F)^2$ in the expression of $\mathcal{I}_c$. By demanding $\mathcal{I}_c$ to be non-negative for these configurations one arrives at the fourth inequality. The first inequality, which places a lower bound on the coefficient of the Skyrme term was previously obtained in [6] from a Lagrangian with π- and ρ-mesons. In the present case it results from combining the third and fourth inequalities.

These conditions (24) show that the Skyrme term and other four-point interactions are essential if the Hamiltonian is to be bounded from below. In an effective theory where the spin-1 fields transform homogeneously they arise as counterterms for the bad behaviour of the vector-meson contributions. This is in sharp contrast to the approach of [3,4], where the same Skyrme term emerges from the exchange of a very heavy ρ-meson.

The constraints we obtain here ensure that the kinetic energies of the specific charge-zero meson configurations considered are bounded from below. Other configurations, including ones with non-zero winding numbers, can also be investigated but we have not found any which lead to more stringent constraints on the couplings. We believe that our results are general for any theory defined by a Lagrangian of the form (8, 9, 20).

5. Discussion

To summarise our results so far: our investigation of classical nonperturbative effects in low-energy chiral theories shows that the constraints (24), relating three- and four-point couplings, must be satisfied for a consistent description of the interactions between pions and spin-1 isovector mesons. We stress that chiral symmetry is implemented nonlinearly in this approach and the vector mesons are naturally assumed to transform homogeneously under chiral rotations. The constraints arise from demanding that the Hamiltonian be bounded from below. They do not depend on phenomenological ideas such as vector dominance.

Before trying to determine phenomenologically the various coupling constants in this effective Lagrangian of pions, ρ's and a_1's, one might ask whether there are any other constraints on them from first principles. For instance another nonperturbative notion that one could invoke in this context is the unitarity of the scattering matrix. This was studied in ref. [10] for the special case of the Lagrangian (8, 9) without the a_1 ($g_2 = g_3 = 0$). Working at tree-level, or leading order in a $1/N_c$ expansion, the authors of ref. [10] found that further local interactions between the pions must be added by hand if the forward elastic $\pi\pi$ scattering amplitude is to obey the Froissart bound [11]. These local interactions compensate for the most divergent contribution produced by ρ-exchange. In the three-flavor case they have the form

$$\begin{aligned}\mathcal{L}^{SU(3)}_{local} = \frac{g_1^2}{8}\Big\{ &< \partial_\mu U \partial^\mu U^\dagger >^2 + 2 < \partial_\mu U^\dagger \partial_\nu U >< \partial^\mu U^\dagger \partial^\nu U > \\ &- 6 < \partial_\mu U^\dagger \partial^\mu U \partial_\nu U^\dagger \partial^\nu U > \Big\},\end{aligned} \tag{25}$$

where U is an $SU(3)$ mapping. Converting this to our notation via the relation $\partial_\mu U U^\dagger = (1/i) u u_\mu u^\dagger$ and using $\tau_a \tau_b = \delta_{ab} + i\epsilon_{abc}\tau_c$ to reduce it to the SU(2) sector, we find that it is

$$\mathcal{L}^{SU(2)}_{local} = \frac{g_1^2}{8} < [u_\mu, u_\nu]^2 > . \tag{26}$$

This is just the Skyrme term, but with a coefficient that is fixed by the three-point coupling g_1. If one works at tree level, as in ref. [10], the a_1 does not contribute to $\pi\pi$ scattering, and so this value for the four-point coupling is also appropriate to our more general $\pi\rho a_1$

theory. Hence imposing unitarity as in ref. [10] leads to saturation of the lower bound on g_4 in (24):

$$g_4 - g_1^2 = 0. \tag{27}$$

Combining this with the final constraint in (24), we obtain a similar relation expressing the implications of unitarity for the couplings of the *axial* meson:

$$g_5 = 2g_1 g_3. \tag{28}$$

This relates the strength of the $a_1 \to \pi\pi\pi$ decay to those of the processes $\rho \to \pi\pi$ and $a_1 \to \rho\pi$.

This saturation of two of our constraints in (24) follows from the assumption that a single vector meson state contributes in the forward $\pi\pi$ scattering amplitude – an extreme version of vector dominance for strong interactions. Realistically one also expects higher-mass vector mesons to contribute; including them in the unitarity argument would require additional terms of the form (6). The coefficient of the Skyrme term would not then be given in terms of the $\rho\pi\pi$ coupling g_1 alone. It would continue to satisfy the first of our inequalities (24), but not the equality (27). As shown in Ref. [10] the value for this coefficient determined assuming ρ-meson dominance agrees well with that from chiral perturbation theory [12]. This suggests that the vector dominance assumption holds to a reasonable accuracy. We speculate that a similar assumption of dominance of a single resonance may also hold in the axial-vector channel, leading to saturation of the remaining constraints in (24). In this case our lagrangian would simplify to:

$$\begin{aligned}\mathcal{L}_{\pi\rho a_1} =& \frac{f^2}{4} < u_\mu u^\mu > + \frac{M_\rho^2}{2} < V_\mu V^\mu > + \frac{M_a^2}{2} < A_\mu A^\mu > \\ & -\frac{1}{4} < \left(V_{\mu\nu} + \frac{i}{\sqrt{2}} \big(g_1 [u_\mu, u_\nu] + g_3 ([A_\mu, u_\nu] - [A_\nu, u_\mu]) \big) \right)^2 > \\ & -\frac{1}{4} < \left(A_{\mu\nu} + \frac{i}{\sqrt{2}} g_2 ([V_\mu, u_\nu] - [V_\nu, u_\mu]) \right)^2 > .\end{aligned} \tag{28}$$

This constitutes an effective lagrangian describing the strong interactions of $\pi\rho a_1$ mesons with a minimal number of free coupling constants. It is amusing to note that in this case the transformation matrix relating the pion time derivative and its momentum still has the form of the original nonlinear sigma model. This new lagrangian is the simplest one compatible

with chiral symmetry and leading to a hamiltonian which is free of pathologies. We believe that it should be regarded as the starting point for any extension of chiral perturbation theory [12] to the resonance region.

As a future prospect, and in connection with baryon physics, let us mention that all past attempts to build topological solitons of the $\pi\rho a_1$ system have failed: the solitons of previously proposed lagrangians have been shown to be generically unstable [13,14]. We would like to stress here that these attempts were only based on massive Yang-Mills or hidden gauge symmetry assumptions. It would now be very interesting to investigate the issue of soliton stability in the alternative framework described here for $\pi\rho a_1$ physics.

Finally it will be important to compare the predictions of the various treatments of ρ and a_1 mesons for processes like $\rho \to \pi\pi\pi\pi$. Accurate measurements of these at DAΦNE [15] could provide a stringent test of effective theories including vector mesons [16].

Acknowledgements

We are grateful to Bachir Moussallam for communication of his results on the asymptotic behaviour of three-point functions in low energy effective lagrangians. DK benefitted from the remarks and encouragements of Robert Vinh Mau during the early stages of this work. MCB is grateful to the SERC for an Advanced Fellowship and DK acknowledges support from the SERC.

Appendix: Construction of H_T

In this note we present details of the derivation of the kinetic part of the secondary Hamiltonian density which appears in eq. (11). We display here the results for the three-point theory defined in section 3 by the Lagrangian $\mathcal{L}^{(2)}_{\pi\rho a_1} + \mathcal{L}^{(3)}_{\pi\rho a_1}$. The structures which appear in the Hamiltonian for the theory with four-point interactions are exactly the same and so the corresponding expression can be obtained by appropriate substitutions of combinations of the couplings, as described in section 4.

We first build the primary Hamiltonian. The conjugate momenta $\vec{\phi},\ \vec{\pi}_i,\ \vec{\chi}_i$ for the fields $\vec{F},\ \vec{V}_i,\ \vec{A}_i$ can be found in the usual way by differentiating the Lagrangian with respect to the time derivatives of the fields. Inverting this relation yields:

$$\begin{aligned}
\dot{\vec{F}} =& \mathcal{A}^{-1}(\vec{\phi} - \mathcal{B}_i\vec{\pi}_i - \mathcal{C}_i\vec{\chi}_i - \vec{\theta}) \\
\dot{\vec{V}}_i =& \frac{-\vec{\pi}_i}{2} + \mathcal{B}_i^T\mathcal{A}^{-1}(\vec{\phi} - \mathcal{B}_j\vec{\pi}_j - \mathcal{C}_j\vec{\chi}_j - \vec{\theta}) + \vec{\zeta}_i^V \\
\dot{\vec{A}}_i =& \frac{-\vec{\chi}_i}{2} + \mathcal{C}_i^T\mathcal{A}^{-1}(\vec{\phi} - \mathcal{B}_j\vec{\pi}_j - \mathcal{C}_j\vec{\chi}_j - \vec{\theta}) + \vec{\zeta}_i^A,
\end{aligned} \tag{A.1}$$

where the script capital letters denote 3×3 matrices acting on isospin vectors and the superscript T denotes transposition. The matrices in these equations are:

$$\begin{aligned}
\mathcal{A} = \mathcal{A}^T =& f^2\mathcal{G} - 4(g_1\mathcal{N}_i - g_3\mathcal{Q}_i^A)^T(g_1\mathcal{N}_i - g_3\mathcal{Q}_i^A) - 4g_2^2(\mathcal{Q}_i^V)^T\mathcal{Q}_i^V \\
\mathcal{B}_i^T =& -\frac{1}{2}\left\{2\mathcal{M}_i^V - \frac{4}{\sqrt{2}}(g_1\mathcal{N}_i - g_3\mathcal{Q}_i^A)\right\} \\
\mathcal{C}_i^T =& -\frac{1}{2}\left\{2\mathcal{M}_i^A + \frac{4}{\sqrt{2}}g_2\mathcal{Q}_i^V\right\},
\end{aligned} \tag{A.2}$$

with the definitions (all indices label isospin, except i and j which we use for space components):

$$\begin{aligned}
(\mathcal{G})_{ab} =& \hat{F}_a\hat{F}_b + \frac{\sin^2 F}{F^2}(\delta_{ab} - \hat{F}_a\hat{F}_b) \\
(\mathcal{M}_i^V)_{ab} =& \frac{\partial(V_{0i})_a}{\partial(\partial_0 F_b)} = -\frac{2\sin^2(F/2)}{F^2}\left[\delta_{ab}(\vec{F}.\vec{V}_i) - F_a(V_i)_b\right] \\
(\mathcal{M}_i^A)_{ab} =& \frac{\partial(A_{0i})_a}{\partial(\partial_0 F_b)} = -\frac{2\sin^2(F/2)}{F^2}\left[\delta_{ab}(\vec{F}.\vec{A}_i) - F_a(A_i)_b\right] \\
(\mathcal{N}_i)_{ab} =& \epsilon_{pqa}\partial_i F_r(\sqrt{\mathcal{G}})_{pb}(\sqrt{\mathcal{G}})_{qr} \\
(\mathcal{Q}_i^V)_{ab} =& -\epsilon_{pqa}(V_i)_p(\sqrt{\mathcal{G}})_{qb} \\
(\mathcal{Q}_i^A)_{ab} =& -\epsilon_{pqa}(A_i)_p(\sqrt{\mathcal{G}})_{qb}.
\end{aligned} \tag{A.3}$$

The isospin vectors in (A.1) are linear functions of the dependent variables $\vec{V}_0$ and $\vec{A}_0$ and are given by:

$$
\begin{aligned}
\theta_k =& 2ig_3(g_1\mathcal{N}_i - g_3\mathcal{Q}_i^A)_{mk}[A_0, u_i]_m - 2ig_2^2(\mathcal{Q}_i^V)_{mk}[V_0, u_i]_m \\
(\zeta_i^V)_k =& (\nabla_i V_0)_k - \frac{ig_3}{\sqrt{2}}[A_0, u_i]_k \\
(\zeta_i^A)_k =& (\nabla_i A_0)_k - \frac{ig_2}{\sqrt{2}}[V_0, u_i]_k.
\end{aligned}
\tag{A.4}
$$

The Hamiltonian is given by the following Legendre transformation:

$$
H = \int d^3x \bigg\{ \vec{\phi}\dot{\vec{F}} - \vec{\pi}_i \dot{\vec{V}}_i - \vec{\chi}_i \dot{\vec{A}}_i - \mathcal{L}^{(2)}_{\pi\rho a_1} - \mathcal{L}^{(3)}_{\pi\rho a_1} \bigg\}. \tag{A.5}
$$

It can be split into two pieces $H = H_T' + H_V$, where the primary "kinetic" energy is H_T' and the "potential" energy is H_V. The form of the latter is given in eq. (10). The primary "kinetic" energy, which is our concern here, contains all the pieces that depend on the conjugate momenta or involve $\vec{V}_0,\ \vec{A}_0$:

$$
\begin{aligned}
H_T' = \int d^3x \bigg\{ & \frac{1}{2}(\vec{\phi} - \vec{\pi}_i\mathcal{B}_i^T - \vec{\chi}_i\mathcal{C}_i^T - \vec{\theta})\mathcal{A}^{-1}(\vec{\phi} - \mathcal{B}_i\vec{\pi}_i - \mathcal{C}_i\vec{\chi}_i - \vec{\theta}) + \frac{\vec{\pi}_i^2}{4} + \frac{\vec{\chi}_i^2}{4} \\
& - \vec{\pi}_i\vec{\zeta}_i^V - \vec{\chi}_i\vec{\zeta}_i^A - M_\rho^2\vec{V}_0^2 - M_a^2\vec{A}_0^2 - \frac{g_3^2}{2}[A_0, u_i]_k^2 - \frac{g_2^2}{2}[V_0, u_i]_k^2 \bigg\}.
\end{aligned}
\tag{A.6}
$$

As can be seen this involves a nontrivial mixing between the pion conjugate momentum and the time components of both vector fields and axial fields. This is in contrast to the minimal $\pi\rho$ case studied in [6] where such a mixing does not occur. This feature complicates the determination of the kinetic part of the Hamiltonian as a function of the independent dynamical variables only, as is needed for proper quantisation.

We wish to eliminate the dependent variables $\vec{V}_0,\ \vec{A}_0$ from the expression for the energy. For this purpose it proves convenient to rewrite the primary H_T' functional in the following suggestive form:

$$
H_T' = \int d^3x \bigg\{ \frac{1}{2}\vec{\Phi}\mathcal{A}^{-1}\vec{\Phi} + \frac{\vec{\pi}_i^2}{4} + \frac{\vec{\chi}_i^2}{4} - \Big[\vec{\Gamma}\vec{\Delta} + \frac{1}{2}\vec{\Delta}\ \mathcal{P}\ \vec{\Delta} \Big] \bigg\}, \tag{A.7}
$$

where we have introduced the notation $\vec{\Phi} = \vec{\phi} - \mathcal{B}_i\vec{\pi}_i - \mathcal{C}_i\vec{\chi}_i$ for simplicity and done some integration by parts so that the gradients of $\vec{V}_0,\ \vec{A}_0$ do not appear in H_T'. The symbols

$\vec{\Delta}$, $\vec{\Gamma}$ denote two-component vectors in the space spanned by $\vec{V}_0$ and $\vec{A}_0$. Their expressions read:

$$\vec{\Gamma} = \begin{pmatrix} \vec{\alpha} \\ \vec{\beta} \end{pmatrix}, \quad \vec{\Delta} = \begin{pmatrix} \vec{V}_0 \\ \vec{A}_0 \end{pmatrix} \tag{A.8}$$

with

$$\begin{aligned} \alpha_k =& -(\nabla_i \pi_i)_k + \frac{i g_2}{\sqrt{2}} [\chi_i, u_i]_k + 4 g_2^2 (\mathcal{Z}^V \mathcal{A}^{-1})_{kp} \Phi_p \\ \beta_k =& -(\nabla_i \chi_i)_k + \frac{i g_3}{\sqrt{2}} [\pi_i, u_i]_k + 4 \Big[(g_3^2 \mathcal{Z}^A - g_3 g_1 \mathcal{Z}^\pi) \mathcal{A}^{-1} \Big]_{kp} \Phi_p \\ \mathcal{Z}^V_{km} =& (\mathcal{Q}^V_j)_{am} \epsilon_{kra} (u_j)_r, \ \mathcal{Z}^A_{km} = (\mathcal{Q}^A_j)_{am} \epsilon_{kra} (u_j)_r, \ \mathcal{Z}^\pi_{km} = (\mathcal{N}_j)_{am} \epsilon_{kra} (u_j)_r. \end{aligned} \tag{A.9}$$

In eq. (A.7), $\mathcal{P}$ is a 2×2 matrix acting on these vectors:

$$\mathcal{P} = \begin{pmatrix} D & E \\ E^T & H \end{pmatrix}. \tag{A.10}$$

The isospin content of the matrix elements D, E, H is:

$$\begin{aligned} D_{km} =& 2 M_\rho^2 \delta_{km} - 4 g_2^2 (\vec{u}_i^2 \delta_{km} - u_i^k u_i^m) - 16 \Big[g_2^2 \mathcal{Z}^V \mathcal{A}^{-1} g_2^2 (\mathcal{Z}^V)^T \Big]_{km} \\ H_{km} =& 2 M_a^2 \delta_{km} - 4 g_3^2 (\vec{u}_i^2 \delta_{km} - u_i^k u_i^m) - 16 \Big[(g_3^2 \mathcal{Z}^A - g_3 g_1 \mathcal{Z}^\pi) \mathcal{A}^{-1} (g_3^2 \mathcal{Z}^A - g_3 g_1 \mathcal{Z}^\pi)^T \Big]_{km} \\ E_{km} =& -16 \Big[g_2^2 \mathcal{Z}^V \mathcal{A}^{-1} (g_3^2 \mathcal{Z}^A - g_3 g_1 \mathcal{Z}^\pi)^T \Big]_{km}. \end{aligned} \tag{A.11}$$

In the canonical formalism the conservation in time of the primary constraints of the theory $\vec{\pi}_0 = \vec{\chi}_0 = 0$ is used in order to eliminate the $\vec{\Delta}$-dependence of the hamiltonian. This conservation law amounts to the vanishing of the Poisson brackets of these momenta with H'_T. This leads to six equations relating the constrained time components of the ρ and the a_1 fields to other fields and their conjugate momenta. In our compact matrix notation these take the following simple form:

$$\left\{ \begin{pmatrix} \vec{\pi}_0 \\ \vec{\chi}_0 \end{pmatrix}, H'_T \right\} = \vec{\Gamma} + \mathcal{P} \vec{\Delta} = 0, \tag{A.12}$$

where $\{\ ,\ \}$ denotes the Poisson bracket. One can check that for $g_2 = g_3 = 0$ this reduces to the covariant form of Gauss's law for massive vector fields [6]: $V_0 = (1/2 M_\rho^2) \nabla_i \pi_i$.

If we suppose the matrix $\mathcal{P}$ to be invertible, $\vec{\Delta}$ can finally be removed from the Hamiltonian using eq. (A.12) to leave us with a functional of the independent dynamical variables $\vec{F}$, $\vec{V}_i$, $\vec{A}_i$, $\vec{\phi}$, $\vec{\pi}_i$, $\vec{\chi}_i$ only. This is the secondary kinetic energy H_T that we study in the text:

$$H_T = \int d^3x \left\{ \frac{1}{2}\vec{\Phi}\mathcal{A}^{-1}\vec{\Phi} + \frac{\vec{\pi}_i^2}{4} + \frac{\vec{\chi}_i^2}{4} + \frac{1}{2}\vec{\Gamma}\mathcal{P}^{-1}\vec{\Gamma} \right\}, \tag{A.13}$$

where the inverse of $\mathcal{P}$ can be written

$$\mathcal{P}^{-1} = \begin{pmatrix} (D - EH^{-1}E^T)^{-1} & -(D - EH^{-1}E^T)^{-1}EH^{-1} \\ -(H - E^TD^{-1}E)^{-1}E^TD^{-1} & (H - E^TD^{-1}E)^{-1} \end{pmatrix}. \tag{A.14}$$

References

[1] S. Weinberg, Phys. Rev. 166 (1968) 1568; I. Gerstein, R. Jackiw, B. Lee and S. Weinberg, Phys. Rev. D3 (1971) 2486; A. Salam and J. Strathdee, Phys. Rev. D2 (1970) 2869.

[2] G. 't Hooft, Nucl. Phys. B79 (1974) 276.

[3] J. J. Sakurai, *Currents and Mesons*, University of Chicago Press, Chicago 1969; J. Schwinger, *Particles and Sources*, Clarendon Press, Oxford, UK, 1969.

[4] M. Bando, T. Kugo and K. Yamawaki, Phys. Rep. 164 (1988) 217.

[5] G. Ecker, J. Gasser, A. Pich and E. de Rafael, Nucl. Phys. B321 (1989) 311.

[6] D. Kalafatis, Phys. Lett. B313 (1993) 115.

[7] S. Coleman, J. Wess and B. Zumino, Phys. Rev. 177 (1969) 2239; C. G. Callan, S. Coleman, J. Wess and B. Zumino, *ibid* 2247.

[8] J. Bijnens, C. Bruno and E. de Rafael, Nucl. Phys. B390 (1993) 501; E. Pallante and R. Petronzio, Nucl. Phys. B396 (1993) 205; E. Pallante and R. Petronzio, Roma preprint ROM2F 93/37.

[9] B. Moussallam, private communication.

[10] G. Ecker, J. Gasser, H. Leutwyler, A. Pich and E. de Rafael, Phys. Lett. B223 (1989) 425.

[11] M. Froissart, Phys. Rev. 123 (1961) 1053; A. Martin, Phys. Rev. 129 (1963) 1432.

[12] J. Gasser and H. Leutwyler, Ann. Phys. (NY) 158 (1984) 142; Nucl. Phys. B250 (1985) 465, 517, 538.

[13] H. Forkel, A. D. Jackson and C. Weiss, Nucl. Phys. A526 (1991) 453.

[14] Z. F. Ezawa and T. Yanagida, Phys. Rev. D33 (1986) 247 ; J. Kunz and D. Masak, Phys. Lett. B179 (1986) 146.

[15] *Proposal for a Phi factory*, Laboratori Nazionali di Frascati-INFN report LNF-90/031 (R) (april 1990).

[16] A. Bramon, A. Grau and G. Pancheri, Phys. Lett. B317 (1993) 190.